Infinite, Highly Connected Digraphs with No Two Arc-Disjoint Spanning Trees

Ron Aharoni
MATHEMATICS DEPARTMENT
TECHNION, HAIFA, ISRAEL

Carsten Thomassen
MATHEMATICAL INSTITUTE
THE TECHNICAL UNIVERSITY OF DENMARK
LYNGBY, DENMARK

ABSTRACT

We present a construction of countably infinite, highly connected graphs and digraphs, which shows that several basic connectivity results on finite graphs, including Edmonds's branching theorem, cannot be extended to the infinite case.

1. INTRODUCTION

The most fundamental graph connectivity result is Menger's theorem. A *cut* in a digraph D is defined as follows: If $V(D)$ is partitioned into sets A and B, then the set E of all arcs from A to B is cut. If $s \in A$ we say that E is a *cut from* s, and if $t \in B$ we say that E is a *cut to* t. Now Menger's theorem can be formulated as follows:

(1) If s and t are vertices in a finite digraph D, then D contains a cut E from s to t and a collection $\mathcal{P}$ of pairwise arc-disjoint directed paths from s to t such that each path in $\mathcal{P}$ contains precisely one arc of E and every arc of E is in a path of $\mathcal{P}$. In particular, $|\mathcal{P}| = |E|$.

A *branching* from s in D is a spanning tree T in D such that all arcs of T are directed away from s. Edmonds' branching theorem can be formulated in a way analogous to (1).

Journal of Graph Theory, Vol. 13, No. 1, 71–74 (1989)
 CCC 0364-9024/89/010071-04$04.00

(2) If s is a vertex in a finite digraph D, then D contains a cut E from s and a collection $\mathcal{T}$ of pairwise arc-disjoint branchings from s such that $|\mathcal{T}| = |E|$.

It is easy to obtain (1) from (2). Let k be the smallest cardinality of a cut from s to t and add, for each vertex, $u \neq t$, $k + 1$ arcs from t to u. Then apply (2) to the resulting digraph. By a similar trick one can derive (1) from the following result of Mader [3]:

(3) If s and t are vertices in a finite $(k + 1)$-arc-connected digraph D, then D contains a directed path P from s to t such that $D - E(P)$ is k-arc-connected.

Mader [3] suggested that (3) might also be true for infinite digraphs.

Nash-Williams [5] proved that every $2m$-edge-connected finite graph admits an m-arc-connected orientation. Combining this with (3) we get the following for k even:

(4) If s and t are vertices in a finite $(k + 2)$-edge-connected graph G, then G contains a path P from s to t such that $G - E(P)$ is k-edge-connected.

For k odd, (4) was proved by Mader [4].

Nash-Williams's orientation theorem combined with (2) also implies the following:

(5) Every finite $2k$-edge-connected graph contains k pairwise edge-disjoint spanning trees.

(5) also follows from the characterization (due to Tutte, Edmonds, and Nash-Williams, see [8]) of the finite graphs having no k edge-disjoint spanning trees. Oxley [6] proved that this characterization does not apply to the infinite case for $k = 2$. However, Oxley's examples do not disprove (5) in the infinite case. The following result in [7] has some analogy with (3) and (4):

(6) Every finite $(k + 3)$-connected graph G contains a cycle C such that $G - V(C)$ is k-connected.

Here we present a construction that shows that none of (2), (3), (4), (5), and (6) are true in the countably infinite case.

2. THE CONSTRUCTION

Theorem. For each natural number $k \geq 2$ there exists a countably infinite graph G containing two vertices s and t such that G has a strongly k-connected orientation D, and such that, for every path P from s to t, $G - E(P)$ is disconnected.

Proof. It is easy to see that there exists a graph H containing vertices $v_1, v_2, \ldots, v_k$ such that the distance between any two of $v_1, \ldots, v_k$ is at least $k + 2$ and such that H has an orientation that is strongly k-connected. (For example, take the disjoint union of many large sets $A_1, A_2, \ldots$ and add all edges between A_i and A_{i+1} for $i = 1, 2, \ldots$.) Now define a sequence of finite graphs $G_1, G_2, \ldots$ recursively as follows: Let $G_1 = H$ and put $V_1' = V(G_1) = V(H)$. Let $s = v_1$, $t = v_2$. Suppose we have already defined G_n ($n \geq 1$) and that the vertex set of G_n can be written as the disjoint union $V_1 \cup V_2 \cup \cdots \cup V_{n-1} \cup V_n'$. Now consider a connected component W of the subgraph of G_n induced by V_n'. We subdivide every edge of W as follows: Let e be any edge of W and let q be the number of distinct paths of length k in W containing e. Now we subdivide e by inserting q "new" vertices of degree 2 on e. We let V_n be the union of V_n' and all the new vertices. Let $e_1, e_2, \ldots, e_k$ be the edges of any path of length k in W and let u_i be one of the new vertices on e_i for $i = 1, 2, \ldots, k$. Then we add a copy of H and we identify v_i and u_i for $i = 1, 2, \ldots, k$. We do this for each path of length k in W and we make the vertex identifications such that all the added copies of H are disjoint. In other words, each new vertex is identified with a vertex in precisely one copy of H. We denote the resulting graph by G_{n+1} and let $V_{n+1}' = V(G_{n+1}) \backslash (V_1 \cup V_2 \cup \cdots \cup V_n)$.

Now let G be the graph with vertex set $V_1 \cup V_2 \cup \cdots$ such that the neighbors of a vertex v in V_n are precisely the neighbors of v in G_{n+1}. Note that we may think of G as composed of copies of subdivisions of H. Since H has a strongly k-connected orientation D' we can use D' to obtain an orientation D_n of G_n and an orientation D of G. It is easy to prove, by induction on n, that each D_n (and hence also D) is strongly k-connected.

Now consider any path P in G from s to t. Let n be the largest number such that P intersects V_n. Then P is a subdivision of a path P' in G_n. Let P'' be a subpath of P' such that the ends of P'' are in V_{n-1} and all intermediate vertices of P'' are in V_n'. Then P'' is a path in a copy of H connecting two vertices in $\{v_1, v_2, \ldots, v_k\}$. Hence P'' minus the ends is a path with at least k edges $e_1, e_2, \ldots, e_k, \ldots$ and with vertices in V_n'. In G_{n+1} there is a copy of H whose vertices $v_1, v_2, \ldots, v_k$ are identified with new vertices on $e_1, e_2, \ldots, e_k$. (In particular, V_{n+1} is nonempty.) That copy of H becomes separated from the rest of G_{n+1} when we delete $E(P)$ from G_{n+1}. It also follows from the construction of G that $G - E(P)$ is disconnected. ∎

The theorem shows that (2), (3), (4), and (5) are false in the infinite case even if the connectivity of the graph or digraph is large and we only seek two trees in (2), (5), or a connected graph after the path deletion in (3) and (4). The construction in the proof of the theorem also shows that (6) is false in the infinite case (even if we replace $G - V(C)$ by $G - E(C)$). We just let H be a k-connected graph of girth at least k containing vertices $v_1, \ldots, v_k$, no two of which are joined by a path of length less than $k + 2$. One quick way to see that such a graph exists is to use the existence of highly chromatic graphs of large girth combined with the result of Mader (see [8]) that a graph of minimum degree at least $4k$ contains a k-connected subgraph. Note that a graph of girth at least k^2 contains k vertices, no two of which are joined by a path of length less than k.

References

[1] R. Aharoni, Menger's theorem for countable graphs, *J. Combinat. Theory Ser. B,* to appear.

[2] J. Edmonds, Edge-disjoint branchings, *Combinatorial Algorithms,* Academic Press, New York (1973) 91–96.

[3] W. Mader, On a property of n-edge-connected digraphs, *Combinatorica* **1** (1981) 385–386.

[4] W. Mader, Paths in graphs, reducing the edge connectivity only by two, *Graphs Combinat.* **1** (1985) 81–89.

[5] C. St. J. A. Nash-Williams, On orientations, connectivity and odd-vertex-pairings in finite graphs, *Can. J. Math.* **12** (1960) 555–567.

[6] J. G. Oxley, On a packing problem for infinite graphs and independence spaces, *J. Combinat. Theory Ser. B* **26** (1979) 123–130.

[7] C. Thomassen, Nonseparating cycles in k-connected graphs, *J. Graph Theory* **5** (1981) 351–354.

[8] C. Thomassen, Configurations in graphs of large minimum degree, connectivity or chromatic number, *Proceedings of the Third International Conference on Combinatorial Mathematics,* Academic Science, New York, to appear.